Theory of Fermionic superfluid with SU(2)×SU(6) symmetry

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(Dated: March 15, 2011)

We study theoretically interspecies Cooper pairing in a fermionic system with SU(2)×SU(6) symmetry. We show that, with suitable unitary transformations, the order parameter for the ground state can be reduced to only two non-vanishing complex components. The ground state has a large degeneracy. We find that while some Goldstone modes have linear dispersions, others are quadratic at low frequencies. We compare our results with the case of SU(N).

I. INTRODUCTION

Higher symmetry groups such as SU(3) play an essential role in our understanding of elementary particle physics[1]. However, in typical condensed matter systems, the internal symmetries are much simpler. Superfluid ^{4}He has no spin, whereas ^{3}He and electrons in solids have only spin 1/2. The situation changes with the advances in cold-atomic gases. We have already seen many examples of bosonic systems with finite spins (≥ 1)[2]. There has also been much attention in fermionic systems with more complex internal structure[3], and more recently, in systems where the symmetry is higher than the usual spin rotational symmetries. Examples include the hidden symmetry in spin 3/2 fermions[4], approximate SU(3) symmetry for ^{6}Li near special external magnetic field values[5],[6–13], and the enlarged symmetry for atoms with finite nuclear but no net electronic spin[14–16].

With no net electronic spin, the hyperfine spin of an atom comes entirely from its nucleus. Since the interaction between atoms mainly arises from their electronic clouds, the interatomic interaction is then independent of the total spins of the atoms involved. (This is in contrast with the more general situation where the scattering length between two atoms in general depends on their total spin. See ref[2],[3]). A very interesting system of this class has been studied recently experimentally, namely a mixture of ^{171}Yb and ^{173}Yb[17]. ^{171}Yb and ^{173}Yb have nuclear spins 1/2 and 5/2 respectively. For the ^{171}Yb and ^{173}Yb mixture under discussion, the intraspecies interaction is weak (the s-wave scattering length is $-0.15nm$ between two ^{171}Yb atoms and $10.55nm$ between two ^{173}Yb atoms). However, there is a rather large inter-species attraction, with the scattering length $a \equiv a_{171-173} \approx -30.6nm$[17]. The Kyoto group has already been able to cool this mixture much below the degeneracy temperature, thus raising the interesting possibility of interspecies Cooper pairing in this system.

Motivated by this, we study a two species fermionic system with interspecies attractive interaction. The weak intraspecies interaction is expected to only slightly modify the quantitative details of the system and will be ignored. We shall take a SU(2) internal symmetry for the first species, and SU(6) for the second one, though our findings are immediately generalizable to SU($2f+1$) with general half-integer f's. The system is expected to undergo interspecies Cooper pairing. (We shall only consider weak attractive interactions, and therefore ignore possibilities of bound states involving three or more particles, *c.f.*[5],[13]). The general order parameter is thus a $2 \times (2f + 1)$ complex valued matrix, since pairing can occur between any internal state of the first species and that of the second species. We determine the structure of the order parameter for the ground state, employing the mean-field approximation. We show that, with a suitable choice of basis, the order parameter for the ground state can be reduced to only two non-vanishing complex components. The ground state for this system has also investigated before[18] in the case where the interaction depends on the total spin F of the two interacting atoms and therefore the Cooper pairs have a definitive spin F. We shall mention briefly the relation of our results to theirs.

The ground state is found to possess a large degeneracy. We thus proceed to find the Goldstone modes of this system. We show that there are $2 \times (2f + 1)$ such modes, and we shall evaluate their dispersion relations. We find 4 linear modes, and $2 \times (2f - 1)$ modes which have quadratic dispersion at very low frequencies, but becoming also linear as slightly higher frequencies. We also determine the physical variables coupling to each of these modes, thus indicating how these modes can be excited experimentally.

Currently, the fermi temperature T_F in the experiment of[17] is around 200nK, thus the product $|k_F a| \approx 0.36$ (here k_F is the fermi wavevector). While this allows a weak-coupling consideration as in here, the transition temperature is unfortunately low ($T_c/T_F \sim e^{-\frac{\pi}{|2k_F a|}} \sim 0.01$). However, we hope that eventually the superfluid state can be reached (perhaps using optical Feshbach resonances to enhance the interaction[17]) and the physics discussed here be studied. Also, we shall see that many of our physical results are more general, and thus would be applicable in case other more favorable related systems can be found.

This paper is organized as follows. In section II we consider the ground state properties, and in section III we discuss the collective modes. Our results have many similarities but some differences with the more studied case of SU(N) fermi superfluids. We therefore compare

our results with this case in Section IV. We conclude in section V.

II. GROUND STATE

In this section we discuss the ground state properties, assuming mean-field theory. This requires that the scattering length between the ^{171}Yb and ^{173}Yb atoms be small compared with the interparticle distances, and also that the transition temperature is small compared with the fermi temperatures. Nevertheless, in below we shall argue that many of our results are qualitatively correct beyond mean-field approximations.

Let us denote the annihilation operators for the ^{171}Yb and ^{173}Yb atoms by $a_{\vec{k},\lambda}$ and $c_{\vec{k},\nu}$ respectively, where $\vec{k}$ is the wavevector and $\lambda = \pm\frac{1}{2}$, $\nu = -f, ..., f$ denotes the internal states. The Hamiltonian H has two parts. The kinetic energy is given by

$$H_K = \sum_{\vec{k},\lambda} \xi_k a_{\vec{k},\lambda}^\dagger a_{\vec{k},\lambda} + \sum_{\vec{k},\nu} \xi_k c_{\vec{k},\nu}^\dagger c_{\vec{k},\nu} \qquad (1)$$

Here $\xi_k \equiv \frac{k^2}{2m} - \mu$, m is the mass of the atoms (We ignore the small mass difference between ^{171}Yb and ^{173}Yb here) For simplicity we shall also confine ourselves to the case where the chemical potentials μ of the two species to be equal (and there are no chemical potential differences among the different hyperfine sublevels)[19]. The interaction H_{int} is given by

$$H_{int} = g \sum_{\vec{k},\vec{k}',\vec{q},\lambda,\nu} a_{\vec{k}_+,\lambda}^\dagger c_{-\vec{k}_-,\nu}^\dagger c_{-\vec{k}'_-,\nu} a_{\vec{k}'_+,\lambda} \qquad (2)$$

where $\vec{k}_\pm \equiv \vec{k} \pm \vec{q}/2$. This is the most general inter-species s-wave interaction obeying SU(2)×SU(6) symmetry. g can be eliminated in favor of the scattering length $a_{171-173}$ but we shall not need this explicit relation here.

Within mean-field theory we can replace H_{int} by an effective interaction

$$H_{int}^{eff} = \sum_{\lambda,\nu} \left\{ \sum_{\vec{k}} \left(\Delta_{\lambda,\nu} a_{\vec{k},\lambda}^\dagger c_{-\vec{k},\nu}^\dagger + \Delta_{\lambda,\nu}^* c_{-\vec{k},\nu} a_{\vec{k},\lambda} \right) - \frac{|\Delta_{\lambda,\nu}|^2}{g} \right\} \qquad (3)$$

where $\Delta_{\lambda,\nu}$ has to satisfy the self-consistent equation

$$\Delta_{\lambda,\nu} = (-g) \sum_{\vec{k}} < a_{\vec{k},\lambda} c_{-\vec{k},\nu} > \qquad (4)$$

Before solving this mean-field Hamiltonian we apply the Ginzburg-Landau (GL) theory. The GL free energy has the form

$$\Omega = \alpha \text{Tr} \left[\boldsymbol{\Delta\Delta}^\dagger \right] + \sum_{l \geq 2} \frac{\beta_l}{l} \text{Tr} \left[(\boldsymbol{\Delta\Delta}^\dagger)^l \right] \qquad (5)$$

Here $\boldsymbol{\Delta}$ is a $2 \times (2f+1)$ matrix with elements $\Delta_{\lambda,\nu}$. α, β_l can easily be evaluated but for here it is sufficient to know that all $\beta_l > 0$, and α is positive above some transition temperature T_c and negative below it. We would like to find the form of $\boldsymbol{\Delta}$ which minimizes Ω below T_c. For this, we notice that since all $\beta_l > 0$, $\boldsymbol{\Delta}$ must be such that these higher order terms are minimized for any given $\text{Tr}\left[\boldsymbol{\Delta\Delta}^\dagger\right]$. Let us denote $D_\lambda \equiv [\sum_\nu |\Delta_{\lambda,\nu}|^2]^{1/2}$. We obtain $\boldsymbol{\Delta\Delta}^\dagger = \frac{D_{1/2}^2 + D_{-1/2}^2}{2} + \mathbf{M}$ where $\mathbf{M}$ is a Hermitian matrix, $\text{Tr}[\mathbf{M}] = 0$, and when expanded as $\mathbf{M} = M_1\sigma_1 + M_2\sigma_2 + M_3\sigma_3$ using Pauli matrices $\sigma_{1,2,3}$, we have $M_3 = (D_{1/2}^2 - D_{-1/2}^2)/2$, $M_1 - iM_2 = \sum_\nu \Delta_{\frac{1}{2},\nu}\Delta_{-\frac{1}{2},\nu}^*$. We thus have $\text{Tr}\left[\boldsymbol{\Delta\Delta}^\dagger\right] = D_{1/2}^2 + D_{-1/2}^2$. We also find easily $\text{Tr}\left[(\boldsymbol{\Delta\Delta}^\dagger)^2\right] = (D_{1/2}^2 + D_{-1/2}^2)^2/2 + \text{Tr}[\mathbf{M}^2]$. Thus, for given $\text{Tr}\left[\boldsymbol{\Delta\Delta}^\dagger\right]$, this fourth order term would be minimized if we choose

$$\sum_\nu \Delta_{\frac{1}{2},\nu}\Delta_{-\frac{1}{2},\nu}^* = 0 \qquad (6)$$

and

$$D_{1/2} = D_{-1/2} \equiv D . \qquad (7)$$

Hence, if we regard $\Delta_{1/2,\nu}$ and $\Delta_{-1/2,\nu}$ each as a (unnormalized) wavefunction of a spin f particle, then eq (6) requires that these two wavefunctions are orthogonal, whereas eq (7) shows that they are of equal magnitude. A possible choice of $\boldsymbol{\Delta}$ satisfying eq (6) is one where all elements $\Delta_{\lambda,\nu}$ vanish except $\Delta_{1/2,1/2}$ and $\Delta_{-1/2,-1/2}$. Eq (7) then requires that their magnitude to be equal. We shall give an alternate explanation of eq (6) and (7) below.[20] Using similar reasoning as above, we can actually see that in fact all $l \geq 2$ terms are minimized by the conditions eq (6) and (7). The free energy then becomes $\Omega = 2(\alpha D^2 + \beta_2 D^4/2 + ...)$, the same as the usual BCS theory for a two-component system except an overall extra factor of 2.

Now we return to the microscopic theory. Defining $\Psi_{\lambda,\nu}$ via $\Delta_{\lambda,\nu} = D_\lambda \Psi_{\lambda,\nu}$ thus $\sum_\nu |\Psi_{\lambda,\nu}|^2 = 1$, the pairing term in H_{int}^{eff} can be written as $\sum_\lambda D_\lambda a_{\vec{k},\lambda}^\dagger \left[\sum_\nu \Psi_{\lambda,\nu} c_{-\vec{k},\nu}^\dagger\right] + h.c.$, and thus can be interpreted as pairing between $a_{\vec{k},\lambda}^\dagger$ and the state $\tilde{c}_{-\vec{k},\lambda}^\dagger \equiv \sum_\nu \Psi_{\lambda,\nu} c_{-\vec{k},\nu}^\dagger$, $\lambda = \pm\frac{1}{2}$. (Here, for convenience of writing, we are simply calling the particles by their corresponding operators). Eq (6) implies that the most favorable state is such that $\tilde{c}_{-\vec{k},1/2}^\dagger$ and $\tilde{c}_{-\vec{k},-1/2}^\dagger$ are orthogonal to each other. This is physical reasonable, as then $a_{\vec{k},1/2}^\dagger$ and $a_{\vec{k},-1/2}^\dagger$ does not have to compete with each other to pair with the c-atoms. With $\tilde{c}_{-\vec{k},\pm 1/2}^\dagger$ already defined above, we can introduce $\tilde{c}_{-\vec{k},\nu}^\dagger$ for $\nu \neq \pm\frac{1}{2}$ to make a complete set, therefore a unitary transformation between $\tilde{c}_{-\vec{k},\nu}^\dagger$ and $c_{-\vec{k},\nu}^\dagger$ operators:

$$\tilde{c}_{-\vec{k},\nu}^\dagger = U_{\nu,\nu'} c_{-\vec{k},\nu'}^\dagger \qquad (8)$$

where $U_{\nu,\nu'} = \Psi_{\lambda=\nu,\nu'}$ for $\nu = \pm\frac{1}{2}$, and $U_{\nu,\nu'}$ for $\nu \neq \pm\frac{1}{2}$ can be arbitrary so long as the matrix $\mathbf{U}$ is unitary. The kinetic energy can be re-written as $K = \sum_{\vec{k},\lambda} \xi_k a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \sum_{\vec{k},\nu} \xi_k \tilde{c}^\dagger_{\vec{k},\nu} \tilde{c}_{\vec{k},\nu}$, and H_{int}^{eff} can now be written as $\sum_\lambda \left[D_\lambda(a^\dagger_{\vec{k},\lambda} \tilde{c}^\dagger_{-\vec{k},\lambda} + \tilde{c}_{-\vec{k},\lambda} a_{\vec{k},\lambda}) - \frac{|D_\lambda|^2}{g} \right]$. Thus the pairing term is of the normal BCS form except the sum over λ. Thus the standard BCS calculations can be immediately applied. The quasiparticle energies are thus $E_{\vec{k},\lambda} = \left[\xi_k^2 + D_\lambda^2\right]^{1/2}$ with D_λ playing the role of the energy gap for the λ species. For the expectation values, we have (for zero temperature, to which we confine ourselves for the rest of the paper) $< a_{\vec{k},\lambda} \tilde{c}_{-\vec{k},\lambda} > = \frac{D_\lambda}{2E_{\vec{k},\lambda}}$, $< a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} > = < \tilde{c}^\dagger_{\vec{k},\lambda} \tilde{c}_{\vec{k},\lambda} > \equiv n^0(E_{k,\lambda}) = \frac{1}{2}\left(1 - \frac{\xi_k}{E_{k,\lambda}}\right)$ etc for $\lambda = \pm\frac{1}{2}$. We also have $< a_{\vec{k},\lambda} \tilde{c}_{-\vec{k},\nu} > = 0$ whenever $\nu \neq \lambda$. Expectation values involving the c operators can be obtained by the inverse transformation. We get $< a_{\vec{k},\lambda} c_{-\vec{k},\nu} > = \sum_{\nu'} U_{\nu',\nu} < a_{\vec{k},\lambda} \tilde{c}_{-\vec{k},\nu'} > = U_{\lambda,\nu} < a_{\vec{k},\lambda} \tilde{c}_{-\vec{k},\lambda} > = \Psi_{\lambda,\nu} \frac{D_\lambda}{2E_{k,\lambda}} = \frac{\Delta_{\lambda,\nu}}{2E_{k,\lambda}}$. The self-consistent equation eq (4) becomes $\Delta_{\lambda,\nu} = (-g) \sum_{\vec{k}} \frac{\Delta_{\lambda,\nu}}{2E_{k,\lambda}}$ and thus, after multiplying by $\Delta^*_{\lambda,\nu}$ and sum over ν, either $D_\lambda = 0$, or

$$1 = (-g) \sum_{\vec{k}} \frac{1}{2(\xi_k^2 + D_\lambda^2)^{1/2}} \tag{9}$$

Thus D_λ obeys the usual BCS gap equation. From the form of the Hamiltonian, it is obvious that the most favorable state would have both $D_{1/2}$ and $D_{-1/2}$ finite, and by eq (9), both attain the usual BCS value and thus equal, consistent with eq (7). The particles $\tilde{c}^\dagger_{\vec{k},\nu}$ for $\nu \neq \pm\frac{1}{2}$ are not involved in pairing. They maintain the normal state energies ξ_k, and there are thus $(2f - 1)$ remaining fermi surfaces, and $< \tilde{c}^\dagger_{\vec{k},\nu} \tilde{c}_{\vec{k},\nu} > = f(\xi_k)$, the fermi function.

The above can be readily generalized to higher symmetries. For example, for $SU(4) \times SU(2f+1)$ with $f \geq 3/2$, then the ground state has an order parameter which, in a suitable basis, can be reduced to pairing only between a_λ and c_λ with the same λ. There are $(2f+1)-4 = (2f-3)$ fermi surfaces remaining normal.

III. COLLECTIVE MODES

A. Dispersion relations: weak pairing limit

The ground state therefore has a very high degeneracy. Any choice of the unitary transformation $\mathbf{U}$ gives identi-

cal ground state energy. The system is thus characterized by a large number of Goldstone modes, with the mode frequency ω vanishing as the wavevector $\vec{q}$ approaches zero. These modes are associated with the fluctuations of the order parameter components $\delta\Delta_{\lambda,\nu}(\vec{q})$ away from their equilibrium values. We shall employ the kinetic equation approach[21] to evaluate their dispersion. This method is equivalent to the random phase approximation in diagrammatic approaches. Though we would employ the weak-coupling approximation, we shall argue that many of our results remain qualitatively valid for general interaction strengths, provided that the broken symmetries for the ground state remain the same as that found within the weak-coupling approximation. For simplicity we shall restrict ourselves to zero temperature.

To simplify our notation we shall drop the tildes on the $\tilde{c}$ operators, or equivalently take a reference state where $\mathbf{U}$ is the identity matrix. Physical variables can be obtained easily by applying the unitary transformation $\mathbf{U}$. We list the different types of modes in turn:

(case 1): Modes corresponding to $\delta\Delta_{\lambda,\nu}$ with $\nu \neq \pm\frac{1}{2}$: For definiteness we consider $\delta\Delta_{1/2,3/2}(\vec{q})$. Besides the mean-field pairing terms in eq (3), we include

$$\delta H = \sum_{\vec{k}} \delta\Delta_{1/2,3/2}(\vec{q}) a^\dagger_{\vec{k}_+,1/2} c^\dagger_{-\vec{k}_-,3/2} + h.c. \tag{10}$$

where $\vec{k}_\pm \equiv \vec{k} \pm \frac{\vec{q}}{2}$. The hermitian conjugate ($h.c.$) term, involving $\delta\Delta^*_{1/2,3/2}$ and other $\delta\Delta_{\lambda,\nu}$ turns out to be decoupled from the equations below. $\delta\Delta_{1/2,3/2}(\vec{q})$ has to obey the self-consistent equation

$$\delta\Delta_{1/2,3/2}(\vec{q}) = (-g) \sum_{\vec{k}} < a_{\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)} \tag{11}$$

where the superscript $^{(1)}$ denotes the first order fluctuation contribution. Its equation of motion can be easily obtained using the hamiltonian $H = H_K + H_{int}^{eff} + \delta H$. We get

$$i\frac{\partial}{\partial t} < a_{\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)} = \left(\xi_{k_+} + \xi_{k_-}\right) < a_{\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)} + \Delta_{1/2,1/2} < c^\dagger_{-\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)}$$

$$+\delta\Delta_{1/2,3/2}(\vec{q})\left(<c^\dagger_{-\vec{k}_-,3/2}c_{-\vec{k}_-,3/2}>^{(0)} - <a_{\vec{k}_+,1/2}a^\dagger_{\vec{k}_+,1/2}>^{(0)}\right) \tag{12}$$

Here the superscript $^{(0)}$ stands for equilibrium expectation values. Thus we need also the equation of motion for $<c^\dagger_{-\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$:

$$i\frac{\partial}{\partial t}<c^\dagger_{-\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)} = -\left(\xi_{k_+}-\xi_{k_-}\right)<c^\dagger_{-\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)} +\Delta^*_{1/2,1/2}<a_{\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$$
$$-\delta\Delta_{1/2,3/2}(\vec{q})<c^\dagger_{-\vec{k}_+,1/2}a^\dagger_{\vec{k}_+,1/2}>^{(0)} \tag{13}$$

obtaining thus a closed set of equations for $<a_{\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$ and $<c^\dagger_{-\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$. Fourier transform and solving for $<a_{\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$ and inserting into eq (11), we get

$$0 = \sum_{\vec{k}}\left\{\left[\frac{\frac{1}{2}\left(1-\frac{\xi_{k_+}}{E_{k_+}}\right)f(\xi_{k_-})}{\omega-\xi_{k_-}+E_{k_+}} - \frac{\frac{1}{2}\left(1+\frac{\xi_{k_+}}{E_{k_+}}\right)(1-f(\xi_{k_-}))}{\omega-\xi_{k_-}-E_{k_+}}\right] - \frac{1}{2E_k}\right\} \tag{14}$$

where we have also used eq (9) to eliminate the coupling constant g. The term in the square bracket is a pair susceptibility. We can also directly obtain it from evaluating the response function of $<a_{\vec{k}_+,1/2}c_{-\vec{k}_-,3/2}>^{(1)}$ to δH. The second term is for adding to the ground state a pair of c and a particles with energies ξ_{k_-} and E_{k_+}. This process occurs only when the final states are available, hence the factor $\frac{1}{2}\left(1+\frac{\xi_{k_+}}{E_{k_+}}\right)(1-f(\xi_{k_-}))$ in the numerator. Similar interpretation applies to the first term which stands for annihilation of the pair. Note that the fermi factor restricts $\xi_{k_-}<0$ for this term, hence $E_{k_+}-\xi_{k_-}=E_{k_+}+|\xi_{k_-}|>0$. We also note that eq (14) converges in the ultraviolet.[22]

One can easily check that, if $\vec{q}=0$, eq (14) is satisfied with $\omega=0$, showing that we indeed has a Goldstone mode. We next search for a solution to eq (14) for small q and ω. For this, we add to eq (14) the vanishing quantity $0=\sum_{\vec{k}}\left(-\frac{f(\xi_{k_-})}{2E_{k_+}}-\frac{1-f(\xi_{k_-})}{2E_{k_+}}+\frac{1}{2E_{k_+}}\right)$, and replace the dummy variable $\vec{k}$ by $\vec{k}_+$, so that $\vec{k}_-\to\vec{k}$. Expanding the resulting equations in q and ω, taking angular average, we obtain

$$0 = A_1\omega + A_2\omega^2 + (\frac{A_1}{2}+B_1+B_2)\frac{q^2}{m} \tag{15}$$

where $A_1 \equiv \sum_{\vec{k}}\frac{1}{2E_k}\left[\frac{1-f(\xi_k)}{(E_k+\xi_k)}-\frac{f(\xi_k)}{(E_k-\xi_k)}\right]$, $A_2 \equiv \sum_{\vec{k}}\frac{1}{2E_k}\left[\frac{1-f(\xi_k)}{(E_k+\xi_k)^2}+\frac{f(\xi_k)}{(E_k-\xi_k)^2}\right]$, $B_1 \equiv -\frac{2}{3}\sum_{\vec{k}}\left(\frac{k^2}{2m}\right)\frac{\xi_k}{2E_k^2}\left[\frac{1-f(\xi_k)}{(E_k+\xi_k)}\left(\frac{1}{E_k+\xi_k}+\frac{1}{E_k}\right)-\frac{f(\xi_k)}{(E_k-\xi_k)}\left(\frac{1}{E_k-\xi_k}+\frac{1}{E_k}\right)\right]$, and $B_2 \equiv \frac{3}{8}\sum_{\vec{k}}\left[-\frac{\frac{\xi_k}{2}+\frac{k^2}{6m}}{E_k^3}+\frac{\xi_k^2\frac{k^2}{2m}}{E_k^5}\right]$

Formally the term $A_2\omega^2$ is small and thus can be dropped in the $\omega\to 0$ limit. However, we shall see that in the weak-coupling regime, the coefficient A_1 is small, and hence this term need to be kept in general beyond some small frequency regime ω^* which we shall define later.

Before we discuss this weak-pairing limit in which we are principally interested, it is instructive to evaluate first the dispersion in the strong pairing limit, where $\mu<0$ and $|\mu|\gg\Delta$ (c.f.[27]). Here Δ stands for the value of $|\Delta_{1/2,1/2}|=|\Delta_{-1/2,-1/2}|$ in equilibrium. In this limit, we get $B_1=-\frac{3}{4}A_1$, while B_2 is negligible. $A_2=A_1/(8|\mu|)$ and hence its contribution is always negligible when $\omega\ll|\mu|$. Hence we obtain $\omega=\frac{q^2}{4m}$, the energy of a free particle of mass $2m$ (see also discussions below).

Now we return to the weak-pairing limit, where $\mu\gg\Delta$. The expression A_1 is explicitly particle-hole asymmetric, i.e., if we approximate the density of states near the Fermi surface by a constant $N(0)$ and replace $\sum_{\vec{k}}\to N(0)\int d\xi$, A would vanish. Hence we must use the more accurate expression $\sum_{\vec{k}}\to\frac{m(2m\mu)^{1/2}}{2\pi^2}\int_{-\mu}^\infty d\xi\left(1+\frac{\xi}{\mu}\right)^{1/2}$. Dividing the region of integration to $|\xi|<\mu$ and $|\xi|>\mu$, one can show that the later is smaller than the former in the $\mu\gg\Delta$ limit. The first contribution to A_1 can be re-written as $\frac{m(2m\mu)^{1/2}}{2\pi^2}\int_0^\mu d\xi\left[\left(1+\frac{\xi}{\mu}\right)^{1/2}-\left(1-\frac{\xi}{\mu}\right)^{1/2}\right]\frac{1}{2\Delta^2}\left(1-\frac{\xi}{(\xi^2+\Delta^2)^{1/2}}\right)$

The quantity in the square bracket can be Taylor expanded, and since $\left(1-\frac{\xi}{(\xi^2+\Delta^2)^{1/2}}\right)\approx\frac{\Delta^2}{2\xi^2}$ for $\xi\gg\Delta$, we find $A_1\approx\frac{m(2m\mu)^{1/2}}{2\pi^2}\frac{1}{2\Delta}\left(\frac{\Delta}{2\mu}\ln\frac{\mu}{\Delta}\right)$ in the $\mu\gg\Delta$ limit.

A_2 is even under particle-hole symmetry. Hence, the dominant contribution in the $|\Delta|\ll\mu$ limit can be approximated as $A_2\approx\frac{m(2m\mu)^{1/2}}{2\pi^2}\int_{-\mu}^\mu d\xi\frac{1}{2E}\frac{1}{(E+\xi)^2}\to\frac{m(2m\mu)^{1/2}}{2\pi^2}\frac{1}{2\Delta^2}$.

For B_1, we substitute $\frac{k^2}{2m} = \xi + \mu$, which generates respectively particle-hole asymmetric and symmetric contributions (note the extra factor of ξ in the definition of B_1), and the latter one is associated with a large coefficient μ. This term gives the dominant contribution to B_1. We obtain $B_1 = -\frac{2}{3}\frac{m(2m\mu)^{1/2}}{2\pi^2}\frac{\mu}{\Delta^2}K$ where K is the dimensionless integral $\Delta^2 \int_0^\mu d\xi \frac{\xi}{E^2}\frac{1}{E+\xi}\left(\frac{1}{E+\xi} + \frac{1}{E}\right) \to \frac{1}{2}$ for $\mu \gg \Delta$. Thus $B_1 \gg A$. We find that $B_2 \ll B_1$ in the $\mu \gg \Delta$ limit. (When using a constant density of states, it turns out that the particle-hole symmetric part of B_2 yields exactly zero, and hence we also need to include the energy dependence of the density of states. B_2 is of order $(\Delta/\mu)^2$ smaller than B_1.) We obtain thus finally the dispersion

$$0 = \tilde{A}_1\omega + \frac{\omega^2}{\Delta} - \tilde{B}\frac{q^2}{m} \tag{16}$$

where $\tilde{A}_1 \equiv \frac{\Delta}{2\mu}\ln\frac{\mu}{\Delta}$, $\tilde{B} \equiv \frac{2}{3}\frac{\mu}{\Delta}$. This is a quadratic equation and can be solved explicitly, but the main results can also be obtained by simply comparing terms in eq (15). When $\frac{\tilde{B}q^2}{m\Delta} \ll \tilde{A}_1^2$, that is, when $v_F q \ll v_F q^* \equiv \Delta(\frac{\Delta}{\mu}\ln\frac{\mu}{\Delta})$, where $v_F \equiv \sqrt{2\mu/m}$ is the fermi velocity, we have

$$\omega = \frac{q^2}{2m}\frac{\frac{4}{3}\frac{\mu}{\Delta}}{\left(\frac{\Delta}{2\mu}\ln\frac{\mu}{\Delta}\right)} \tag{17}$$

The frequency is thus quadratic in q, but with a coefficient much larger than $1/m$. Alternatively, we can also write

$$\omega = (q\xi_0)^2\Delta\left[\frac{\frac{2}{3}\pi^2}{\frac{\Delta}{\mu}\ln\frac{\mu}{\Delta}}\right] \tag{18}$$

where $\xi_0 \equiv \frac{v_F}{\pi\Delta}$ is a measure of the zero temperature coherence length. We note that the factor within the square bracket [ ] is much larger than 1. If $q \gg q^*$ however, the dispersion becomes linear, with

$$\omega = \sqrt{\frac{\tilde{B}\Delta}{m}}q = \frac{v_F}{\sqrt{3}}q \tag{19}$$

The transition between these two regime occurs at $q \approx q^*$, where $\omega \approx \omega^* \equiv \Delta\frac{\Delta}{2\mu}\ln\frac{\mu}{\Delta}$.

The existence of this quadratic mode at small q and ω can be understood from gauge symmetry arguments. The equations of motion such as eq (12) and (13) must remain valid under arbitrary gauge transformations (we leave out the subscripts $\vec{k}$ to simplify the writing here) $a_\lambda \to a_\lambda e^{i\theta_\lambda}$ and $c_\nu \to c_\nu e^{i\phi_\nu}$. Hence, the equation of motion for $\delta\Delta_{1/2,3/2}$ must decouple from those for other $\delta\Delta_{\lambda,\nu}$ and also all $\delta\Delta_{\lambda,\nu}^*$'s. For example $\delta\Delta_{1/2,3/2}^*$ transforms differently from $\delta\Delta_{1/2,3/2}$ under a gauge transformation of $c_{3/2}$, and this difference cannot be compensated by any combinations of $\Delta_{1/2,1/2}$ and $\Delta_{-1/2,-1/2}$. It also cannot couple to $\delta\Delta_{-1/2,3/2}$: for example $\Delta_{-1/2,-1/2}^*\delta\Delta_{-1/2,3/2}$ and $\Delta_{1/2,1/2}^*\delta\Delta_{1/2,3/2}$ transform in the same way under transformations of $a_{1/2}$ and $a_{-1/2}$, but they do not do so under transformations of $c_{\pm 1/2}$. We thus obtain a scalar (not the determinant of a matrix, c.f. (case 2) and (case 3) below) equation which relates ω and q (such as eq (14)). The small ω and q expansion of this equation

must therefore be of the form $0 = A_1\omega + B\frac{q^2}{m}$, where A_1 does not vanish unless there is another symmetry (here particle-hole), and the expansion in q begins with q^2 since our system is spatially inversion symmetric[23,24]. Therefore, though our specific formulas assumed weak-coupling, the quadratic dispersion at small q and ω is more general, provided that the broken symmetries remain the same as those found within our weak-coupling theory. The frequency scale ω^* unfortunately is small in the weak-coupling regime. However, we also note that ω^* and q^* increase with Δ/μ, thus the frequency and momentum range where would have this quadratic mode would therer increase with the strength of the attractive interaction.

As seen from eq (12) and (13), this mode couples to "spin-flip" of the c-particles. [This coupling is allowed because, e.g., in eq (12), $< a_{1/2}c_{3/2} >$ and $\Delta_{1/2,1/2} < c_{1/2}^\dagger c_{3/2} >$ transform in the same way under gauge transformations]. The order parameter mode under discussion therefore can be excited by Raman pulses which interconvert $\nu = 1/2$ and $\nu = 3/2$ hyperfine sublevels. Bragg scattering experiments have already been performed in fermi gases[29]. Though that experiment does not involve "spin-flips", it seems that a generalization of the method there can also observe spin-waves and hence the order parameter collective modes here.

Similar discussion applies to all $\nu \neq \pm 1/2$. There are thus $2 \times (2f - 1)$ such modes. The mode labeled by $\delta\Delta_{\lambda,\nu}$ couples to the observable $< c_{-\vec{k}_+,\lambda}^\dagger c_{-\vec{k}_-,\nu} >$, a generalized "spin-density".

One can also consider $\delta\Delta_{\lambda,\nu}^*$ with $\nu \neq \pm 1/2$. These are just the complex conjugates of the modes discussed above, (with frequencies opposite sign[25]) and are not new physical modes.

(case 2): $\delta\Delta_{\lambda,\nu}(\vec{q})$ with $\nu = \lambda$: We can proceed as above. This variable couples to $\delta\Delta_{\lambda,\nu}^*(-\vec{q})$. The algebra is identical with the ordinary BCS case for pairing between two species, and hence we would not display the details of our calculations here. We just remind the readers that, due to the coupling between $\delta\Delta_{\lambda,\nu}(\vec{q})$ and

$\delta\Delta^*_{\lambda,\nu}(\vec{q})$, the dispersion is obtained by setting the determinant of a matrix to be zero[26] (c.f., (case 1) above). In the weak-pairing limit we obtained the mode frequency of the Anderson-Bogoliubov mode[28] $\omega = \frac{v_F}{\sqrt{3}}q$. This mode can also be interpreted in a similar manner as in the two component case. (In the strong-pairing limit we obtain again the Bogoliubov mode for bound boson pair, similar to[27], but we shall not go into that here). These modes couple to the densities fluctuations $< c^\dagger_{-\vec{k}_+,\lambda}c_{-\vec{k}_-,\lambda} >^{(1)}$ and $< a_{-\vec{k}_+,\lambda}a^\dagger_{-\vec{k}_-,\lambda} >^{(1)}$. There are two such modes, one for each choice for λ.

(case 3) $\delta\Delta_{\lambda,-\lambda}(\vec{q})$: we find that it couples with $\delta\Delta^*_{-\lambda,\lambda}(-\vec{q})$ [$\Delta^*_{-\lambda,-\lambda}\delta\Delta_{\lambda,-\lambda}$ and $\Delta_{\lambda,\lambda}\delta\Delta^*_{-\lambda,\lambda}$ transform under the same way under gauge transformations.]. The equation of motion is analogous to (case 2). Thus again we have linear modes with $\omega = \frac{v_F}{\sqrt{3}}q$. These modes cou-

ple to the (spin) densities $< c^\dagger_{-\vec{k}_+,\lambda}c_{-\vec{k}_-,-\lambda} >^{(1)}$ and $< a_{\vec{k}_+,\lambda}a^\dagger_{\vec{k}_-,-\lambda} >^{(1)}$. There are two such modes, again one for each choice for λ.

The two modes in (case 2) can also be viewed as one "density" mode and one "longitudinal spin" mode (though the "density" mode involves only the total densities of the $\lambda = \nu = \pm 1/2$ particles and the "longitudinal" mode involves the differences between them, and the $\nu \neq \pm 1/2$ ones are not involved). The longitudinal spin mode and the two "transverse" spin modes in (case 3) give altogether three linear spin modes with $\omega = \frac{v_F}{\sqrt{3}}q$.

We had not discussed the spin density fluctuations corresponding to $< c^\dagger_{-\vec{k}_+,\nu}c_{-\vec{k}_-,\nu'} >^{(1)}$ for $\nu,\nu' \neq \pm 1/2$. They do not couple to the order parameter fluctuations. Within our approximation they are simply independent particle-hole pairs as in the normal state.

B. bosonic limit

For strong attractive interactions, our formalism above may not apply due to the appearance of multi-particle bound states. However, to gain better understanding of some of our results above, it is instructive to consider this limit assuming we only have tightly bound pairs between a and c particles. We would like to illuminate on the counting of the collective modes and the existence of quadratic versus linear modes. Readers who find these points already clear are invited to skip this subsection.

In this limit the system can be described by bosonic fields $\psi_{\lambda,\nu}$, corresponding to the bound state between a_λ and c_ν. It is simple to construct a theory for a bosonic condensate of this system. The Hamiltonian $H = H_K + H_{int}$ can be written as the sum of the kinetic part

$$H_K = \int_{\vec{r}} \sum_{\lambda,\nu} \left[\frac{\nabla\psi^\dagger_{\lambda,\nu}\psi_{\lambda,\nu}}{2m_b} - \mu_b \psi^\dagger_{\lambda,\nu}\psi_{\lambda,\nu} \right] \tag{20}$$

where m_b denotes the mass of the atom-pair and μ_b denotes the chemical potential, and the interaction, the most general form of which obeying SU(2)×SU(6) symmetry reads

$$H_{int} = \int_{\vec{r}} \frac{\tilde{g}}{8}[\delta_{\lambda_1,\lambda_3}\delta_{\lambda_2,\lambda_4} + \delta_{\lambda_1,\lambda_4}\delta_{\lambda_2,\lambda_3}][\delta_{\nu_1,\nu_3}\delta_{\nu_2,\nu_4} + \delta_{\nu_1,\nu_4}\delta_{\nu_2,\nu_3}]\psi^\dagger_{\lambda_1,\nu_1}\psi^\dagger_{\lambda_2,\nu_2}\psi_{\lambda_3,\nu_3}\psi_{\lambda_4,\nu_4} . \tag{21}$$

For the ground state, we replace the operators $\psi_{\lambda,\nu}$ by c-numbers $\Psi_{\lambda,\nu}$. The resulting energy reads

$$\begin{aligned} E &= -\mu_b \sum_{\lambda,\nu} |\Psi_{\lambda,\nu}|^2 + \frac{\tilde{g}}{4} \sum_{\lambda,\nu,\lambda',\nu'} \left\{ \Psi^*_{\lambda,\nu}\Psi^*_{\lambda',\nu'}\Psi_{\lambda',\nu'}\Psi_{\lambda,\nu} + \Psi^*_{\lambda,\nu}\Psi^*_{\lambda',\nu'}\Psi_{\lambda',\nu}\Psi_{\lambda,\nu'} \right\} \\ &= -\mu_b \text{Tr}\left[\boldsymbol{\Psi\Psi}^\dagger\right] + \frac{\tilde{g}}{4}\left\{ \left(\text{Tr}\left[\boldsymbol{\Psi\Psi}^\dagger\right]\right)^2 + \text{Tr}\left[\boldsymbol{\Psi\Psi}^\dagger\boldsymbol{\Psi\Psi}^\dagger\right] \right\} \end{aligned} \tag{22}$$

where $\boldsymbol{\Psi}$ is a $2 \times (2f+1)$ matrix with entries $\Psi_{\lambda,\nu}$. This energy can be minimized with exactly the same procedure as in sec II. We obtain $\sum_\nu \Psi_{1/2,\nu}\Psi^*_{-1/2,\nu} = 0$ and $\sum_\nu |\Psi_{1/2,\nu}|^2 = \sum_\nu |\Psi_{-1/2,\nu}|^2$ (c.f. eq (6) and (7)). One possible possible solution to these two equation is $\Psi_{1/2,1/2} = \Psi_{-1/2,-1/2} = \Psi$ but with all other components zero. Minimizing E, we get $\mu_b = \frac{3\tilde{g}}{2}|\Psi|^2$. Again we

have many degenerate ground states.

The collective modes are also in one-to-one correspondence with the ones we found above in the weak-pairing regime. They can be found by standard Bogoliubov transformation. We just state the results. The fluctuation $\delta\Psi_{\lambda,\nu}$ for $\nu \neq \pm 1/2$ has dispersion $\omega = \frac{q^2}{2m_b}$, corresponding to free particles. There are $2 \times (2f-1)$ of

these modes. ($\delta\Psi^*_{\lambda,\nu}$ has $\omega = -\frac{q^2}{2m_b}$ but simply correspond to removal of a boson and are not new modes). These are analogous to (*case 1*) discussed above. The four variables $\delta\Psi_{\lambda,\lambda}$ and $\delta\Psi^*_{\lambda,\lambda}$ with $\lambda = \pm 1/2$ are coupled (the equation determining the relation between ω and q is then again obtained by setting the determinant of a matrix to zero.). These yield two Goldstone modes, one with velocity $\omega = cq$ ($\omega = c_s q$) corresponding to in (out-of-)phase oscillations between the $\lambda = 1/2$ and the $\lambda = -1/2$ components. ($\delta\Psi_{1/2,1/2} = +(-)\delta\Psi_{-1/2,-1/2}$ and $\delta\Psi^*_{1/2,1/2} = +(-)\delta\Psi^*_{-1/2,-1/2}$). We obtain $c_s = \frac{1}{\sqrt{3}}c$ and $c^2 = \mu_b/m_b$. These are the analogous to the modes in (*case 2*). Finally, the four variables $\delta\Psi_{\lambda,-\lambda}$ and $\delta\Psi^*_{\lambda,-\lambda}$ with $\lambda = \pm 1/2$ yield two other Goldstone modes again with $\omega = c_s q$. These are the analog of (*case 3*). We have thus in total one "density" mode and three "spin" modes as discussed above. In the present case however the "density" modes and "spin" modes have different velocities due to interactions among bosons. The finding that the "density" and "spin" modes have the same velocities (as well as that they are given by $v_F/\sqrt{3}$) in the weak-coupling limit in the last subsection is the result of the approximation employed there, and is not expected to hold in general.

IV.   COMPARISON WITH SU(N)

We now compare our results with BCS pairing in SU(N) models , where there are N species of fermions $a_{1,...,N}$ with interspecies interaction which is invariant under $SU(N)$ transformations. We have seen that, for our $SU(2) \times SU(6)$ system with interspecies pairing, for the *ground state*, the pairing order parameter reduces to, with a suitable choice of basis, one where each a-species pairs only with one unique c-species. We have seen that this is a consequence of eq (6). The results for SU(N) are similar[6,7] but there is one important difference. For SU(N), the order parameter $\boldsymbol{\Delta}$ is an antisymmetric $N \times N$ matrix (there is no such restriction for our $SU(2) \times SU(6)$ case). Under a unitary transformation $\mathbf{U}$, it transforms as $\boldsymbol{\Delta} \rightarrow \mathbf{U}\boldsymbol{\Delta}\mathbf{U^t}$ where the superscript t denotes the transpose. Hence, with a suitable $\mathbf{U}$, $\boldsymbol{\Delta}$ can always be transformed[30] to one where all entries $\Delta_{\lambda_1,\lambda_2}$ vanish except

$$\Delta_{12} = -\Delta_{21} \qquad \Delta_{34} = -\Delta_{43} \qquad ... , \qquad (23)$$

that is,

$$\Delta_{13} = \Delta_{14} = ... = \Delta_{23} = ... = 0 \qquad (24)$$

so that 1 only pairs with 2, 3 only pairs with 4 etc[31]. However, we emphasize that the origin of eqs (23) and (24) is very different from the $SU(2) \times SU(6)$ case. The possibility of writing $\boldsymbol{\Delta}$ in the form of Eqs (23) and (24) is purely a consequence of the fact that $\boldsymbol{\Delta}$ is antisymmetric[30], which in turn only requires fermionic

anticommutation relations, and therefore holds for *arbitrary* states (including excited states) for the system. For our $SU(2) \times SU(6)$ system, eq (6) needs not hold other than the ground state.

Now we compare the collective modes. Consider first $N = 3$. In the gauge where the only finite component is Δ_{12}, it can easily be shown that $\delta\Delta_{13}$ obeys the same equations of motion as eq (12) and (13) with $a_{1/2} \rightarrow a_1$, $c_{1/2} \rightarrow a_2$, $c_{3/2} \rightarrow a_3$. Hence the dispersion found there also applies. That $\delta\Delta_{13}$ decouples from other order parameter modes can also be seen by gauge invariance arguments as in sec III. It can be easily seen that $\delta\Delta_{23}$ yields yet one more mode with the same dispersion as $\delta\Delta_{13}$. On the other hand, $\delta\Delta_{12}$ and $\delta\Delta^*_{12}$ are coupled together, and they generate one linear mode as in *case 2* in sec III. Hence in total, we have one linear mode, and two modes with quadratic dispersions for small ω and q.

Our results for the number of linear and quadratic modes agree with Ref[8]. (note that they have an alternative but related argument for the existence of quadratic modes). However, the precise dispersion of the quadratic mode is different. They obtain a dispersion same as a free particle of mass m. Our eq (14) is the same as their (A1) at zero temperature (except $\vec{k} \rightarrow \vec{k} - \frac{\vec{q}}{2}$, see however[22]), so we believe that they have made a subsequent algebraic error (the contributions from our B_1 and B_2 terms seem to be missing). That their result is unreasonable was also pointed out in Ref[10] (see also[23]). If particle-hole asymmetric terms are ignored, we saw that the quadratic modes become linear with velocity $v_F/\sqrt{3}$, in agreement with Ref[10], resulting in therefore three linear modes in total. Ref[10] however did not provide the full dispersion relation in the presence of particle-hole asymmetric contributions. Also, we do not understand their claim that, when particle-hole asymmetric terms are included as we have done here, "two of the massless modes split into a massless mode with quadratic dispersion and a massive one". We have seen that $\delta\Delta_{13}$ and $\delta\Delta_{23}$ yield two independent quadratic modes. These two modes would not couple to each other, nor to other order parameter modes by gauge symmetry, and thus each must yield one quadratic mode. Since this is due to gauge symmetry, this result should hold even beyond the weak-coupling approximation we have employed. Lastly, Ref[7] has counted the number of modes differently from here (and therefore also Ref[8]). They regard $\delta\Delta^*_{13}$ and $\delta\Delta^*_{23}$ as giving two other modes in addition to $\delta\Delta_{13}$ and $\delta\Delta_{23}$, and they therefore counted in total five Goldstone modes. As remarked before, $\delta\Delta^*_{13}$ and $\delta\Delta_{13}$ just correspond to annihilation and creation of the same excitation, so it seems natural not to count them separately (see subsection III B). Also, Ref[7] has performed a numerical calculation of the dispersion for the mode $\delta\Delta_{13}$. Their numerical result seems only to show a linear dispersion, likely due to the small Δ/μ values chosen there.

The above discussions can readily generalized to larger N. For example, for SU(5), there are four quadratic modes. With choice of order parameter as in eq (23) and

(24), they are $\delta\Delta_{\lambda,5}$ for $\lambda = 1,2,3,4$. No such modes are expected for N being even. For SU(4), there are six linear but no quadratic modes.

V. CONCLUSION

In this paper, we have considered some superfluid properties of an SU(2)×SU(6) system with interspecies pairing, motivated by the system studied experimentally in ref[17]. We considered both the ground state and collective excitations. Some properties are dramatically different from two-component systems. There are in particular collective mode excitations with quadratic dispersions at low frequencies. Many of our results found are generally applicable to systems with interspecies pairing with high symmetries.

VI. ACKNOWLEDGEMENT

[1] H. Georgi, "Lie Algebras in Particle Physics", Reading, Masschusetts, Perseus Books, 1999.
[2] For a recent review, see, M. Ueda and Y. Kawaguchi, arXiv:1001.2072
[3] T. L. Ho and S. K. Yip, Phys. Rev. Lett. **82**, 247 (1999); S. K. Yip and T. L. Ho, Phys. Rev. A, **59**, 4653 (1999)
[4] C. Wu, Mod. Phys. Lett. B **20**, 1707 (2006); P. Lecheminant, P. Azaria, E. Boulet, S. Capponi,, G. Roux and S. R. White, Int. J. Mod. Phys. E **17**, 2110 (2008); K. Rodriguez, A. Argüelles, M. Colomé-Tatché, T. Vekua and L. Santos, Phys. Rev. Lett. **105**, 050402 (2010)
[5] T. B. Ottenstein et al, Phys. Rev. Lett. **101**, 203202 (2008); J. H. Huckans, J. R. Williams, E. L. Hazlett, R. W. Stites, and K. M. OHara , *ibid*, **102**, 165302 (2009); J. R. Williams, E. L. Hazlett, J. H. Huckans, R. W. Stites, Y. Zhang, and K. M. OHara, *ibid* **103**, 130404 (2009)
[6] A. G. K. Modawi and A. J. Leggett, J. Low Temp. Phys. **109**, 625 (1997)
[7] C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. **92**, 170403 (2004); Phys. Rev. B. **70**, 094521 (2004)
[8] Lianyi He, Meng Jin and Pengfei Zhang, Phys. Rev. A **74**, 033604 (2006)
[9] R. W. Cherng, G. Refael and E. Demler, Phys. Rev. Lett. **99**, 130406 (2007);
[10] G. Catelani and E. A. Yuzbashyan, Phys. Rev. A **78**, 033615 (2008)
[11] J. P. Martikainen, J. J. Kinnunen, P. Törmä and C. J. Pethick *ibid*, **103**, 260403 (2009)
[12] T. Ozawa and G. Baym, Phys. Rev. A **82**, 063615 (2010)
[13] A. Rapp, G. Zaránd, C. Honerkamp and W. Hofstetter, Phys. Rev. Lett.**98**, 160405 (2007); H. Zhai, Phys. Rev. A **75**, 031603 (2007); X.-J. Liu, Hui Hu and P. Drummond, *ibid*, **77**, 013622 (2008) X. W. Guan, M. T. Batchelor, C. Lee and H.-Q. Zhou, Phys. Rev. Lett. **100**, 200401 (2008); P. F. Bedaque and J. P. D'Incao, Ann. Phys. **324**, 1763 (2009);
[14] T. Fukuhara, Y. Takasu, M. Kumakura, Y. Takahashi, Phys. Rev. Lett. **98**, 030401 (2007)
[15] B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, T. C. Killian, Phys. Rev. Lett. **105**, 030402 (2010); M. K. Tey, S. Stellmer, R. Grimm and F. Schreck, Phys. Rev. A **82**, 011608 (2010).
[16] M. A. Cazalilla, A. F. Ho and M. Ueda, New J. Phys. **11**, 103303 (2009); M. Hermele, V. Gurarie and A. M. Rey Phys. Rev. Lett. **103**, 135301 (2009) A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin and A. M. Rey, Nature Physics, **6**, 289 (2010); T. A. Tóth, A. Läuchli, F. Mila and K. Penc, Phys. Rev. Lett. **105**, 265301 (2010);
[17] S. Taie, Y. Takasu, S. Sugawa, R. Yamazaki, T. Tsujimoto, R. Murakami, and Y. Takahashi, Phys. Rev. Lett. **105**, 190401 (2010)
[18] D. B. M. Dickerscheid, Y. Kawaguchi and M. Ueda, Phys. Rev. A **77**, 053605 (2008)
[19] This would correspond to equal populations in all hyperfine levels of ^{171}Yb and ^{173}Yb in the non-interacting limit. However, the populations need not be equal in the presence of a finite order order parameter, *c.f.*[8,9,12]).
[20] Eq (6) and (7) are applicable also to the case where the pairing occurs in a fixed total spin F channel where $F = f + \frac{1}{2}$ or $F = f - \frac{1}{2}$. In this case, we can write $\Delta_{\lambda,\nu} = \sum_M \Delta_{FM} < \frac{1}{2}f, FM|\frac{1}{2}f,\lambda,\nu >$ where $< \frac{1}{2}f, FM|\frac{1}{2}f,\lambda,\nu >$ is the Clebsch-Gordan coefficent of combining the states $|\frac{1}{2},\lambda >$ and $|f,\nu >$ to $|FM >$. Note that F is not summed. Substituting this into eq (6) and (7) and using the values for the Clebsch-Gordan coefficients, we recover the results of[18] that the state is "non-magnetic".
[21] D. Vollhardt and P. Wölfle, "The Superfluid Phases of Helium 3", Taylor and Francis, London, 1990
[22] Note that we used $\vec{k} \pm \vec{q}/2$ for the operators a and c. This guarantees that there would not be terms which behave as $\vec{k}\cdot\vec{q}/k^4$ for $k \to \infty$ which would converge only conditionally.
[23] The term $(\frac{A_1}{2} + B_1 + B_2)\frac{q^2}{m}$ in eq (15) can be understood as the (negative) of a gradient energy in the presence of $\delta\Delta_{1/2,3/2}(\vec{q})$, where we have the extra terms in the Hamiltonian given by $\delta H = \sum_{\vec{k}}\left[\delta\Delta_{1/2,3/2}(\vec{q})a^\dagger_{\vec{k}_+,1/2}c^\dagger_{-\vec{k}_-,3/2} + \delta\Delta^*_{1/2,3/2}(\vec{q})c_{\vec{k}_-,3/2}a_{-\vec{k}_+,1/2}\right] - \frac{|\delta\Delta_{1/2,3/2}(\vec{q})|^2}{g}$. The change of energy δE of the system to second order in $\delta\Delta_{1/2,3/2}(\vec{q})$, can be evaluated by perturbation theory. We get $\delta E = \kappa|\delta\Delta_{1/2,3/2}(\vec{q})|^2$ where κ is simply negative of the expression in the curly brackets in eq (14) with $\omega = 0$.
[24] An alternative way to understand eq (15) is to note that one must have a continuity equation for the spin density $\delta n_s \equiv \sum_{\vec{k}} < c^\dagger_{-\vec{k}_+,1/2}c_{-\vec{k}_-,3/2} >^{(1)}$ in the form $\omega\delta n_s - \vec{q}\cdot\vec{J}_s = 0$, where $\vec{J}_s$ is a spin current. Indeed, this

equation can be obtained by summing eq (13) over all $\vec{k}$. $-\vec{q} \cdot \vec{J}_s$ can be written as, making use of the gap equation eq (11), $-\vec{q} \cdot \vec{J}_s = \sum_{\vec{k}} (\xi_{k_+} - \xi_{k_-}) < c^\dagger_{-\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)}$ $+\Delta^*_{1/2,1/2} \delta\Delta_{1/2,3/2}(\vec{q}) \sum_{\vec{k}} \left[\frac{1}{2E_{k_+}} - \frac{1}{2E_k} \right]$ where the last term arises from the modification of the current operator in the presence of δH. The distribution function $< c^\dagger_{-\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)}$ can be evaluated from eq (12) and (13) or directly from perturbation theory. We have $< c^\dagger_{-\vec{k}_+,1/2} c_{-\vec{k}_-,3/2} >^{(1)} =$ $\frac{1}{2E_{k_+}} \left[\frac{1-f(\xi_{k_-})}{E_{k_+}+\xi_{k_-}-\omega} - \frac{f(\xi_{k_-})}{E_{k_+}-\xi_{k_-}+\omega} \right] \Delta^*_{1/2,1/2} \delta\Delta_{1/2,3/2}(\vec{q})$. Expanding in small q and ω, we obtain $\delta n_s = [A_1 + A_2\omega + ...]\Delta^*_{1/2,1/2}\delta\Delta_{1/2,3/2}(\vec{q})$ and $-\vec{q} \cdot \vec{J}_s = [(\frac{A_1}{2} + B_1 + B_2)\frac{q^2}{m} + ...]\Delta^*_{1/2,1/2}\delta\Delta_{1/2,3/2}(\vec{q})$ where ... denote terms of higher order. The continuity equation then implies eq (15). Hence A_1 represents the static susceptibility of the spin density δn_s in response to the perturbation $\delta\Delta_{1/2,3/2}(\vec{q})$. Note that $-\vec{q} \cdot \vec{J}_s$ is therefore related to the gradient energy discussed in[23].

[25] The dispersion is given by eq (15) with $\omega \rightarrow -\omega$.
[26] This equation has the form[27]

Here $M, B'_1, B'_2, A'_1, A'_2, C'_1$ are coefficients and ... denotes higher order terms in q and/or ω.

[27] J. R. Engelbrecht, M. Randeria and C. A. R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997)

[28] P. W. Anderson, Phys. Rev. **112**, 1900 (1958); N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov, "A New Method in the Theory of Superconductivity", Consultant Bureau, New York, 1959.

[29] G. Veeravalli, E. Kuhnle, P. Dyke, and C. J. Vale, Phys. Rev. Lett. 101, 250403 (2008)

[30] L. K. Hua, Am. J. Math. **66**, 470 (1944); D. C. Youla, Cand. J. Math. **13**, 694 (1961); Serge Lang, in *Algebra* (Chapter XIV, §9), Addision Wesley, Reading, Masschusetts (1965).

[31] Eq (23) and (24) imply that there exists at least one normal fermi surface if N is odd[6,7]. Using eq (23) and (24) in the Ginzburg-Landau free energy functional or weak-coupling gap equation as in section II, we immediately obtain $|\Delta_{12}| = |\Delta_{34}| = ... = D$ (*c.f.* eq (7) and (9)). For the case of $SU(4)$, the above results together then imply that, in any general basis, $|\Delta_{12}| = |\Delta_{34}|$, $|\Delta_{13}| = |\Delta_{24}|$, $|\Delta_{14}| = |\Delta_{23}|$, a result stated in[7].

$$\begin{vmatrix} M + B'_1 q^2 + A'_1\omega + A'_2\omega^2 + ... & M + B'_2 q^2 + C'_1\omega + ... \\ M + B'_2 q^2 - C'_1\omega + ... & M + B'_1 q^2 - A'_1\omega + A'_2\omega^2 + ... \end{vmatrix} = 0$$